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# Connection between static electromagnetoelasticity and anisotropic elasticity and its applications

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**Abstract**

Connection between electromagnetoelasticity and anisotropic elasticity is explored in the state space setting. In the absence of electric charges and currents, the basic equations of static electromagnetoelasticity are formulated into a state equation and an output equation, which bear a remarkable resemblance to the corresponding equations of elasticity. Accordingly, the solutions for various steady-state problems of electromagnetoelasticity can be determined in parallel to their elastic counterparts. For illustration, the generalized plane problems are treated within the context. Exact solutions for the electromagnetoelastic fields in a half-space subjected to line loads and in an infinite plate with an elliptic notch under extension are determined in a simple way.


*Keywords:* Electromagnetoelasticity; Anisotropic elasticity; Generalized plane problems; State space formalism

## 1. Introduction

When a conducting media under the combined actions of mechanical and electromagnetic fields, there occur various interacting effects in the media (Nye, 1957; Solymar, 1984; Kong, 1990). If the mechanical responses are of primary interest, the problem may be formulated on the basis of Maxwell's equations of electromagnetism and equations of elasticity in which the Lorentz force due to the electromagnetic effects is included, together with the constitutive relations that characterize the material properties of the media (Nowacki, 1975; Parton and Kudryavtsev, 1988). Determination of analytic solutions for the mechanical field interacting with the time varying phenomena of electromagnetism poses a formidable problem, if

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not out of the question. In the present study, we confine ourselves to the static responses of the media in steady-state electromagnetic and mechanical fields. This class of electromagnetoelastic problems is relatively simple because all the field variables are independent of time. Even so, analytic solutions are not easy to obtain in view that the equations of anisotropic elasticity are complicated enough in themselves; adding the electromagnetic equations and field variables makes it more difficult. The usual solution approach is to extend the method used in obtaining the elastic solution to include the electromagnetic effects (e.g., Wang and Shen, 2002; Gao et al., 2003; Chen and Lee, 2003; Wang et al., 2003; Chen et al., 2004; Pan and Han, 2005). The extensions were often made in a straightforward and ad hoc manner.

In solving the problem of electromagnetoelasticity, it is cumbersome and unwieldy to work with the individual equations and variables, especially when the medium is arbitrarily anisotropic. If a certain link between electromagnetoelasticity and anisotropic elasticity can be found, it is conceivable that the solution for an electromagnetoelastic problem may be determined in parallel to its elastic counterpart. Questions arise as to whether there exists any meaningful link between them. While it is difficult to find such a link following the usual approach, a useful connection emerges upon formulating the basic equations of static electromagnetoelasticity in the state space setting. A key step to bring out the connection is to form the state vector by grouping the field variables using matrix notation and partitioning the constitutive matrix accordingly. Upon defining the generalized displacements vector and the generalized stress vectors, the 3D equations of static electromagnetoelasticity can be cast neatly into two matrix equations in which all the field variables and material constants are represented by only three vectors and four matrices. Consequently, there is no need to deal with the equations and variables individually, and it becomes easy to derive the state equation and the output equation for the problem. More importantly, the matrix equations bear a remarkable resemblance to their elastic counterparts (Tarn, 2002a,b,c), differing only in the sizes and entities of the corresponding matrices. This enables us to determine the solutions for various stationary problems of electromagnetoelasticity by a simple extension of the solutions for the corresponding problems of elasticity. In many cases, the solutions can be obtained directly by carrying over the elastic solutions. For illustration, we treat the generalized plane problems within the context. Exact solutions for the electromagnetoelastic field in a half-space subjected to 2D loads and in an infinite plate with an elliptical notch under extension are determined in a simple way.

## 2. State space formulation

### 2.1. Basic equations

The constitutive relations for linear, anisotropic electromagnetoelastic media can be expressed as

$$\sigma_{ij} = c_{ijkl}\varepsilon_{kl} - e_{kij}E_k - q_{kij}H_k, \tag{1}$$

$$D_i = e_{ijk}\varepsilon_{jk} + \epsilon_{ik}E_k + d_{ik}H_k, \tag{2}$$

$$B_i = q_{ijk}\varepsilon_{jk} + d_{ik}E_k + \mu_{ik}H_k, \tag{3}$$

where $\sigma_{ij}$ and $\varepsilon_{ij}$ are the stress and strain tensors, $D_i$ and $E_i$ the electric displacement and electric field strength, $B_i$ and $H_i$ the magnetic flux density and magnetic field strength, $c_{ijkl}$, $e_{kij}$, $q_{kij}$, $\epsilon_{ik}$, $d_{ik}$, and $\mu_{ik}$ the elastic constants, the piezoelectric constants, the piezomagnetic constants, the dielectric constants, the electromagnetic coupling constants, and the magnetic permeability constants, respectively. These constants possess the symmetric properties

$$c_{ijkl} = c_{jikl} = c_{ijlk} = c_{klij}, \quad e_{kij} = e_{kji}, \quad q_{kij} = q_{kji}, \quad \epsilon_{ik} = \epsilon_{ki}, \quad d_{ik} = d_{ki}, \quad \mu_{ik} = \mu_{ki}. \tag{4}$$

Eqs. (1)–(3) encompass the constitutive equations for linear elasticity, piezoelectricity, and electromagnetism of anisotropic media. Setting $e_{ijk} = q_{ijk} = 0$, Eq. (1) becomes the constitutive equations of anisotropic elastic materials without polarization and magnetization; Eqs. (2) and (3) the constitutive equations of electromagnetic materials (Kong, 1990). Setting $q_{ijk} = d_{ik} = 0$, Eqs. (1) and (2) become the constitutive equations of piezoelectric materials (Nye, 1957); Eq. (3) the constitutive equations of magnetic materials.

In the absence of electric currents and electric charges, Maxwell's equations for static problems of electromagnetism reduce to

$$B_{i,i} = 0, \quad D_{i,i} = 0, \tag{5}$$

$$E_i = -\phi_{,i}, \quad H_i = -\varphi_{,i}, \tag{6}$$

where a comma indicates differentiation with respect to the suffix variable, $\phi$ and $\varphi$ are the electric potential and the magnetic potential, respectively.

The strain–displacement relations are

$$\varepsilon_{ij} = (u_{i,j} + u_{j,i})/2, \tag{7}$$

where $u_i$ are the displacement components.

The equations of equilibrium are

$$\sigma_{ij,j} + F_i = 0, \tag{8}$$

where $F_i$ denotes the body force.

For a 3D problem of a generally anisotropic material, there are 72 independent material constants and 29 field variables (three displacements, six strains, six stresses, three electric displacements, three electric fields, three magnetic fields, and three magnetic fluxes, three electromagnetic potentials) in total. Eqs. (1)–(3) and (5)–(8) constitute the 29 equations necessary for a unique solution. Clearly, it would be very cumbersome to deal with the field variables and material constants individually.

### 2.2. Equations in matrix forms

To facilitate the ensuing formulation, we use the contracted notation (Nye, 1957) to express Eqs. (1)–(3), in the matrix form

$$\begin{bmatrix} \sigma_{11} \\ \sigma_{22} \\ \sigma_{33} \\ \sigma_{23} \\ \sigma_{13} \\ \sigma_{12} \\ D_1 \\ D_2 \\ D_3 \\ B_1 \\ B_2 \\ B_3 \end{bmatrix} = \begin{bmatrix} c_{11} & c_{12} & c_{13} & c_{14} & c_{15} & c_{16} & e_{11} & e_{21} & e_{31} & q_{11} & q_{21} & q_{31} \\ c_{12} & c_{22} & c_{23} & c_{24} & c_{25} & c_{26} & e_{12} & e_{22} & e_{32} & q_{12} & q_{22} & q_{32} \\ c_{13} & c_{23} & c_{33} & c_{34} & c_{35} & c_{36} & e_{13} & e_{23} & e_{33} & q_{13} & q_{23} & q_{33} \\ c_{14} & c_{24} & c_{34} & c_{44} & c_{45} & c_{46} & e_{14} & e_{24} & e_{34} & q_{14} & q_{24} & q_{34} \\ c_{15} & c_{25} & c_{35} & c_{45} & c_{55} & c_{56} & e_{15} & e_{25} & e_{35} & q_{15} & q_{25} & q_{35} \\ c_{16} & c_{26} & c_{36} & c_{46} & c_{56} & c_{66} & e_{16} & e_{26} & e_{36} & q_{16} & q_{26} & q_{36} \\ e_{11} & e_{12} & e_{13} & e_{14} & e_{15} & e_{16} & -\epsilon_{11} & -\epsilon_{12} & -\epsilon_{13} & -d_{11} & -d_{12} & -d_{13} \\ e_{21} & e_{22} & e_{23} & e_{24} & e_{26} & e_{26} & -\epsilon_{12} & -\epsilon_{22} & -\epsilon_{23} & -d_{12} & -d_{22} & -d_{23} \\ e_{31} & e_{32} & e_{33} & e_{34} & e_{35} & e_{36} & -\epsilon_{13} & -\epsilon_{23} & -\epsilon_{33} & -d_{13} & -d_{23} & -d_{33} \\ q_{11} & q_{12} & q_{13} & q_{14} & q_{15} & q_{16} & -d_{11} & -d_{12} & -d_{13} & -\mu_{11} & -\mu_{12} & -\mu_{13} \\ q_{21} & q_{22} & q_{23} & q_{24} & q_{25} & q_{26} & -d_{12} & -d_{22} & -d_{23} & -\mu_{12} & -\mu_{22} & -\mu_{23} \\ q_{31} & q_{32} & q_{33} & q_{34} & q_{35} & q_{36} & -d_{13} & -d_{23} & -d_{33} & -\mu_{13} & -\mu_{23} & -\mu_{33} \end{bmatrix} \begin{bmatrix} \varepsilon_{11} \\ \varepsilon_{22} \\ \varepsilon_{33} \\ 2\varepsilon_{23} \\ 2\varepsilon_{13} \\ 2\varepsilon_{12} \\ -E_1 \\ -E_2 \\ -E_3 \\ -H_1 \\ -H_2 \\ -H_3 \end{bmatrix}. \tag{9}$$

The formulation is greatly simplified if the field variables are grouped properly. For the problems in Cartesian coordinates, if the $x_2$ axis is pointed in the thickness direction, the traction vector $(\sigma_{12}, \sigma_{22}, \sigma_{23})$, the normal electric displacement $D_2$, and the normal magnetic flux $B_2$ are directly associated with the surface boundary conditions and the interfacial continuity conditions on the planes $x_2 = \text{constant}$. With this in mind, we group the field variables into two parts: one consists of the components associated with the subscript 2, another consists of the remaining components. This way of grouping is particularly useful for problems of layered media and laminated systems with the planes $x_2 = \text{constant}$ being the interfaces and the boundary surfaces.

Upon partitioning the constitutive matrix in accordance with the grouping, Eq. (9) can be expressed in a concise form

$$\begin{bmatrix} \boldsymbol{\tau}_1 \\ \boldsymbol{\tau}_2 \end{bmatrix} = \begin{bmatrix} \mathbf{C}_{11} & \mathbf{C}_{12} \\ \mathbf{C}_{12}^{\mathrm{T}} & \mathbf{C}_{22} \end{bmatrix} \begin{bmatrix} \boldsymbol{\gamma}_1 \\ \boldsymbol{\gamma}_2 \end{bmatrix}, \tag{10}$$

where $\boldsymbol{\tau}_1$ and $\boldsymbol{\tau}_2$ may be regarded as the generalized stress vectors, $\boldsymbol{\gamma}_1$ and $\boldsymbol{\gamma}_2$ as the corresponding generalized strains, and

$$\boldsymbol{\tau}_1 = [\sigma_{13} \quad \sigma_{11} \quad \sigma_{33} \quad D_1 \quad D_3 \quad B_1 \quad B_3]^{\mathrm{T}}, \quad \boldsymbol{\tau}_2 = [\sigma_{12} \quad \sigma_{22} \quad \sigma_{23} \quad D_2 \quad B_2]^{\mathrm{T}},$$

$$\boldsymbol{\gamma}_1 = [2\varepsilon_{13} \quad \varepsilon_{11} \quad \varepsilon_{33} \quad -E_1 \quad -E_3 \quad -H_1 \quad -H_3]^{\mathrm{T}}, \quad \boldsymbol{\gamma}_2 = [2\varepsilon_{12} \quad \varepsilon_{22} \quad 2\varepsilon_{23} \quad -E_2 \quad -H_2]^{\mathrm{T}},$$

$$\mathbf{C}_{11} = \begin{bmatrix} c_{55} & c_{15} & c_{35} & e_{15} & e_{35} & q_{15} & q_{35} \\ c_{15} & c_{11} & c_{13} & e_{11} & e_{31} & q_{11} & q_{31} \\ c_{35} & c_{13} & c_{33} & e_{13} & e_{33} & q_{13} & q_{33} \\ e_{15} & e_{11} & e_{13} & -\epsilon_{11} & -\epsilon_{13} & -d_{11} & -d_{13} \\ e_{35} & e_{31} & e_{33} & -\epsilon_{13} & -\epsilon_{33} & -d_{13} & -d_{33} \\ q_{15} & q_{11} & q_{13} & -d_{11} & -d_{13} & -\mu_{11} & -\mu_{13} \\ q_{35} & q_{31} & q_{33} & -d_{13} & -d_{33} & -\mu_{13} & -\mu_{33} \end{bmatrix},$$

$$\mathbf{C}_{12} = \begin{bmatrix} c_{56} & c_{25} & c_{45} & e_{25} & q_{25} \\ c_{16} & c_{12} & c_{14} & e_{21} & q_{21} \\ c_{36} & c_{23} & c_{34} & e_{23} & q_{23} \\ e_{16} & e_{12} & e_{14} & -\epsilon_{12} & -d_{12} \\ e_{36} & e_{32} & e_{34} & -\epsilon_{23} & -d_{23} \\ q_{16} & q_{12} & q_{14} & -d_{12} & -\mu_{12} \\ q_{36} & q_{32} & q_{34} & -d_{23} & -\mu_{23} \end{bmatrix}, \quad \mathbf{C}_{22} = \begin{bmatrix} c_{66} & c_{26} & c_{46} & e_{26} & q_{26} \\ c_{26} & c_{22} & c_{24} & e_{22} & q_{22} \\ c_{46} & c_{24} & c_{44} & e_{24} & q_{24} \\ e_{26} & e_{22} & e_{24} & -\epsilon_{22} & -d_{22} \\ q_{26} & q_{22} & q_{24} & -d_{22} & -\mu_{22} \end{bmatrix}.$$

The strain–displacement relations can be cast into

$$\begin{bmatrix} \boldsymbol{\gamma}_1 \\ \boldsymbol{\gamma}_2 \end{bmatrix} = \begin{bmatrix} \mathbf{L}_1\mathbf{u} \\ \mathbf{L}_2\mathbf{u} \end{bmatrix} + \partial_2 \begin{bmatrix} \mathbf{0} \\ \mathbf{u} \end{bmatrix}, \tag{11}$$

where $\partial_i$ stands for the partial derivative with respect to $x_i$, $\mathbf{u}$ may be regarded as the generalized displacement vector, and

$$\mathbf{u} = [u_1 \quad u_2 \quad u_3 \quad \phi \quad \varphi]^{\mathrm{T}}, \quad \mathbf{L}_1 = \mathbf{K}_1\partial_1 + \mathbf{K}_2\partial_3, \quad \mathbf{L}_2 = \mathbf{K}_3\partial_1 + \mathbf{K}_4\partial_3,$$

$$\mathbf{K}_1 = \begin{bmatrix} 0 & 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}, \quad \mathbf{K}_2 = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix},$$

$$\mathbf{K}_3 = \begin{bmatrix} 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}, \quad \mathbf{K}_4 = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}.$$

Substituting Eq. (11) into Eq. (10), we have

$$\begin{bmatrix} \boldsymbol{\tau}_1 \\ \boldsymbol{\tau}_2 \end{bmatrix} = \begin{bmatrix} (\mathbf{C}_{11}\mathbf{L}_1 + \mathbf{C}_{12}\mathbf{L}_2)\mathbf{u} \\ (\mathbf{C}_{12}^{\mathrm{T}}\mathbf{L}_1 + \mathbf{C}_{22}\mathbf{L}_2)\mathbf{u} \end{bmatrix} + \begin{bmatrix} \mathbf{C}_{12}\partial_2\mathbf{u} \\ \mathbf{C}_{22}\partial_2\mathbf{u} \end{bmatrix}. \tag{12}$$

With the $\mathbf{K}_i$ matrices and the differential operators $\mathbf{L}_1$ and $\mathbf{L}_2$, we can express Eqs. (5) and (8) in a single matrix equation

$$\partial_2\boldsymbol{\tau}_2 + \mathbf{L}_1^{\mathrm{T}}\boldsymbol{\tau}_1 + \mathbf{L}_2^{\mathrm{T}}\boldsymbol{\tau}_2 + \mathbf{F} = \mathbf{0}, \tag{13}$$

where

$$\mathbf{F} = [F_1 \quad F_2 \quad F_3 \quad 0 \quad 0]^{\mathrm{T}}.$$

Eqs. (12) and (13) embrace the 3D equations of static electromagnetoelasticity in full. With the basic equations so expressed, the individual field variables and material constants are no longer in view—they are replaced by $\mathbf{u}$, $\boldsymbol{\tau}_1$, $\boldsymbol{\tau}_2$, and $\mathbf{C}_{\alpha\beta}$, $(\alpha, \beta = 1, 2)$. Consequently, we only need to work with three vectors and four matrices instead of the individual variables and material constants. Moreover, the matrix equations bear a remarkable resemblance to their elastic and piezoelectric counterparts (Tarn, 2002a,c), differing only in the sizes and entities of the corresponding matrices. As a result, connections between static electromagnetoelasticity and anisotropic elasticity emerge.

With reference to the state space formalism for anisotropic elasticity and piezothermoelasticity (Tarn, 2002a,b,c), choosing the generalized displacement vector $\mathbf{u}$ and the generalized stress vector $\boldsymbol{\tau}_2$ to form the state vector, we can write down immediately the state equation and output equation for static electromagnetoelasticity as follows:

$$\frac{\partial}{\partial x_2}\begin{bmatrix} \mathbf{u} \\ \boldsymbol{\tau}_2 \end{bmatrix} = \begin{bmatrix} \mathbf{D}_{11} & \mathbf{C}_{22}^{-1} \\ \mathbf{D}_{21} & \mathbf{D}_{11}^{\mathrm{T}} \end{bmatrix}\begin{bmatrix} \mathbf{u} \\ \boldsymbol{\tau}_2 \end{bmatrix} - \begin{bmatrix} \mathbf{0} \\ \mathbf{F} \end{bmatrix}, \tag{14}$$

$$\boldsymbol{\tau}_1 = \begin{bmatrix} \widetilde{\mathbf{C}}_{11}\mathbf{L}_1 & \mathbf{C}_{12}\mathbf{C}_{22}^{-1} \end{bmatrix}\begin{bmatrix} \mathbf{u} \\ \boldsymbol{\tau}_2 \end{bmatrix}, \tag{15}$$

where

$$\mathbf{D}_{11} = -\mathbf{C}_{22}^{-1}\mathbf{C}_{12}^{\mathrm{T}}\mathbf{L}_1 - \mathbf{L}_2, \quad \mathbf{D}_{21} = -\mathbf{L}_1^{\mathrm{T}}\widetilde{\mathbf{C}}_{11}\mathbf{L}_1, \quad \widetilde{\mathbf{C}}_{11} = \mathbf{C}_{11} - \mathbf{C}_{12}\mathbf{C}_{22}^{-1}\mathbf{C}_{12}^{\mathrm{T}}.$$

For a specific problem being solved the task is to determine the state vector that satisfies the state equation together with the boundary conditions. Once the state vector is determined, all the unknown field variables follow from the output equation. We are now in a better position to deal with various problems.

## 3. Generalized plane problems

When a body of uniform cross-section is subjected to surface loadings that do not vary in the $x_3$-axis, all the field variables are independent of $x_3$ except for the displacement components. This class of problems is referred to as the generalized plane problem in anisotropic elasticity (Lekhnitskii, 1981). The general expressions of the displacement components take the form

$$\begin{bmatrix} u_1 \\ u_2 \\ u_3 \end{bmatrix} = \begin{bmatrix} u(x_1,x_2) \\ v(x_1,x_2) \\ w(x_1,x_2) \end{bmatrix} + \varepsilon \begin{bmatrix} 0 \\ 0 \\ x_3 \end{bmatrix} + \vartheta \begin{bmatrix} -x_2x_3 \\ x_1x_3 \\ 0 \end{bmatrix} + b_1 \begin{bmatrix} -x_3^2/2 \\ 0 \\ x_1x_3 \end{bmatrix} + b_2 \begin{bmatrix} 0 \\ -x_3^2/2 \\ x_2x_3 \end{bmatrix}, \tag{16}$$

where $u$, $v$, and $w$ are unknown functions of $x_1$ and $x_2$, the rigid body displacements have been excluded, the constant $\varepsilon$ is a uniform extension, $\vartheta$ is associated with the curvature due to twisting, $b_1$ and $b_2$ are associated with the curvatures due to bending.

For a combined loading at the ends, the constants $\varepsilon$, $\vartheta$, $b_1$, and $b_2$ can be determined from the end conditions that require the stress resultants over the cross section $A$ reduce to an axial force $P_3$, a torque $M_3$, and bi-axial bending moments $M_1$ and $M_2$:

$$\int_A (\mathbf{H}_1\boldsymbol{\tau}_1 + \mathbf{H}_2\boldsymbol{\tau}_2)\mathrm{d}x_1\,\mathrm{d}x_2 = \mathbf{P}, \tag{17}$$

where

$$\mathbf{H}_1 = \begin{bmatrix} 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & x_2 & 0 & 0 & 0 & 0 \\ 0 & 0 & -x_1 & 0 & 0 & 0 & 0 \\ -x_2 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}, \quad \mathbf{H}_2 = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & x_1 & 0 & 0 \end{bmatrix}, \quad \mathbf{P} = \begin{bmatrix} P_3 \\ M_1 \\ M_2 \\ M_3 \end{bmatrix}.$$

Alternatively, we may solve various cases of simple loadings and obtain the solution for the combined loading by superposition. As there is a one to one correspondence between $\varepsilon$, $\vartheta$, $b_1$, $b_2$ and the prescribed end loads, these constants may be regarded as known a priori.

On substituting Eq. (16) in Eqs. (14) and (15), the state equation and the output equation become

$$\frac{\partial}{\partial x_2}\begin{bmatrix} \tilde{\mathbf{u}} \\ \boldsymbol{\tau}_2 \end{bmatrix} = \begin{bmatrix} -\mathbf{C}_{22}^{-1}\mathbf{A}_1\partial_1 & \mathbf{C}_{22}^{-1} \\ -\mathbf{A}_2\partial_{11} & -\mathbf{A}_1^{\mathrm{T}}\mathbf{C}_{22}^{-1}\partial_1 \end{bmatrix}\begin{bmatrix} \tilde{\mathbf{u}} \\ \boldsymbol{\tau}_2 \end{bmatrix} - \begin{bmatrix} \mathbf{p}_1 \\ \mathbf{p}_2 \end{bmatrix} - \begin{bmatrix} \mathbf{0} \\ \mathbf{f} \end{bmatrix}, \tag{18}$$

$$\boldsymbol{\tau}_1 = \begin{bmatrix} \widetilde{\mathbf{C}}_{11}\mathbf{K}_1\partial_1 & \mathbf{C}_{12}\mathbf{C}_{22}^{-1} \end{bmatrix}\begin{bmatrix} \tilde{\mathbf{u}} \\ \boldsymbol{\tau}_2 \end{bmatrix} + \widetilde{\mathbf{C}}_{11}[(\varepsilon + b_1x_1 + b_2x_2)\mathbf{k}_1 - \vartheta x_2\mathbf{k}_2], \tag{19}$$

where

$$\tilde{\mathbf{u}} = [u \quad v \quad w \quad \phi \quad \varphi]^{\mathrm{T}}, \quad \mathbf{f} = [F_1 \quad F_2 \quad 0 \quad 0 \quad 0]^{\mathrm{T}},$$
$$\mathbf{A}_1 = \mathbf{C}_{12}^{\mathrm{T}}\mathbf{K}_1 + \mathbf{C}_{22}\mathbf{K}_3, \quad \mathbf{A}_2 = \mathbf{K}_1^{\mathrm{T}}\widetilde{\mathbf{C}}_{11}\mathbf{K}_1,$$
$$\mathbf{p}_1 = \mathbf{C}_{22}^{-1}\mathbf{C}_{12}^{\mathrm{T}}[(\varepsilon + b_1x_1 + b_2x_2)\mathbf{k}_1 - \vartheta x_2\mathbf{k}_2] + \vartheta x_1\mathbf{k}_3,$$
$$\mathbf{p}_2 = b_1[\tilde{c}_{13} \quad 0 \quad \tilde{c}_{35} \quad \tilde{e}_{13} \quad \tilde{q}_{13}]^{\mathrm{T}}, \quad \mathbf{k}_1 = [0 \quad 0 \quad 1 \quad 0 \quad 0 \quad 0 \quad 0]^{\mathrm{T}},$$
$$\mathbf{k}_2 = [1 \quad 0 \quad 0 \quad 0 \quad 0 \quad 0 \quad 0]^{\mathrm{T}}, \quad \mathbf{k}_3 = [0 \quad 0 \quad 1 \quad 0 \quad 0]^{\mathrm{T}}.$$

We seek the homogeneous solution of Eq. (18) of the form

$$\tilde{\mathbf{u}} = \mathbf{U}F(z), \quad \boldsymbol{\tau}_2 = \mathbf{S}F'(z), \tag{20}$$

where $\mathbf{U}$ and $\mathbf{S}$ are constant vectors, each has five components; $F(z)$ is an unknown function of complex variables, $F'(z) = \mathrm{d}F(z)/\mathrm{d}z$, $z = x_1 + px_2$, $p$ is a constant parameter to be determined.

Substituting Eq. (20) in Eq. (18) yields the eigen relation

$$\begin{bmatrix} -\mathbf{C}_{22}^{-1}\mathbf{A}_1 & \mathbf{C}_{22}^{-1} \\ -\mathbf{A}_2 & -\mathbf{A}_1^{\mathrm{T}}\mathbf{C}_{22}^{-1} \end{bmatrix} \begin{bmatrix} \mathbf{U} \\ \mathbf{S} \end{bmatrix} = p \begin{bmatrix} \mathbf{U} \\ \mathbf{S} \end{bmatrix}, \tag{21}$$

where $p$ is the eigenvalue, $[\mathbf{U}, \mathbf{S}]^{\mathrm{T}}$, is the eigenvector. For a given material the eigenvalues and the associated eigenvectors can be determined using *Mathematica* or MATLAB.

Expressing $\mathbf{S}$ in terms of $\mathbf{U}$ using Eq. $(21)_1$, we have

$$\mathbf{S} = (\mathbf{A}_1 + p\mathbf{C}_{22})\mathbf{U}. \tag{22}$$

Substituting Eq. (22) in Eq. $(21)_2$ leads to

$$[\mathbf{A}_3 + p(\mathbf{A}_1 + \mathbf{A}_1^{\mathrm{T}}) + p^2\mathbf{C}_{22}]\mathbf{U} = \mathbf{0}, \tag{23}$$

where

$$\mathbf{A}_3 = \mathbf{K}_1^{\mathrm{T}}\mathbf{C}_{11}\mathbf{K}_1 + \mathbf{K}_1^{\mathrm{T}}\mathbf{C}_{12}\mathbf{K}_3 + \mathbf{K}_3^{\mathrm{T}}\mathbf{C}_{12}^{\mathrm{T}}\mathbf{K}_1 + \mathbf{K}_3^{\mathrm{T}}\mathbf{C}_{22}\mathbf{K}_3.$$

Non-trivial solution to Eq. (23) exists if and only if the determinant of the coefficient matrix vanishes

$$|\mathbf{A}_3 + p(\mathbf{A}_1 + \mathbf{A}_1^{\mathrm{T}}) + p^2\mathbf{C}_{22}| = 0. \tag{24}$$

By setting $e_{ij} = q_{ij} = 0$, Eq. (24) reduces to the sextic equation and Eq. (21) to the eigen relation in the Stroh formalism of anisotropic elasticity (Ting, 1996). Analogous to the formulation in anisotropic elasticity, it can be shown that the $p$ cannot be real by virtue of the positive-definiteness of the free energy, and there are five pairs of complex conjugate $p$. Denote the eigenvalues and the associated eigenvectors by

$$p_k = a_k + ib_k, \quad p_{k+5} = \bar{p}_k = a_k - ib_k, \quad b_k > 0, \tag{25}$$

$$\mathbf{U}_{k+5} = \overline{\mathbf{U}}_k, \quad \mathbf{S}_{k+5} = \overline{\mathbf{S}}_k, \quad k = 1 - 5, \tag{26}$$

where $i$ is the imaginary number, $a_k$ and $b_k$ are real.

There follows:

$$\tilde{\mathbf{u}} = 2\mathrm{Re}\left\{\sum_{k=1}^{5} \mathbf{U}_k F_k(z_k)\right\}, \tag{27}$$

$$\boldsymbol{\tau}_1 = 2\mathrm{Re}\left\{\sum_{k=1}^{5} (\mathbf{A}_4 + p_k\mathbf{C}_{12})\mathbf{U}_k F_k'(z_k)\right\}, \tag{28}$$

$$\boldsymbol{\tau}_2 = 2\mathrm{Re}\left\{\sum_{k=1}^{5} (\mathbf{A}_1 + p_k\mathbf{C}_{22})\mathbf{U}_k F_k'(z_k)\right\}, \tag{29}$$

where $\mathbf{U}_k$ is the eigenvector associated with the eigenvalue $p_k$, $z_k = x_1 + p_k x_2$, and

$$\mathbf{A}_4 = \mathbf{C}_{11}\mathbf{K}_1 + \mathbf{C}_{12}\mathbf{K}_3.$$

The particular solution of Eq. (18) is

$$\tilde{\mathbf{u}} = \mathbf{a}_1 x_1^2/2 + \mathbf{a}_2 x_1 x_2 + \mathbf{a}_3 x_2^2/2, \tag{30}$$

$$\boldsymbol{\tau}_2 = \varepsilon \mathbf{C}_{12}^{\mathrm{T}} \mathbf{k}_1, \tag{31}$$

$$\boldsymbol{\tau}_1 = \varepsilon \mathbf{C}_{11} \mathbf{k}_1 - \vartheta \widetilde{\mathbf{C}}_{11} \mathbf{k}_2 x_2 + \widetilde{\mathbf{C}}_{11}[(\mathbf{K}_1 \mathbf{a}_1 + b_1 \mathbf{k}_1) x_1 + (\mathbf{K}_1 \mathbf{a}_2 + b_2 \mathbf{k}_1) x_2], \tag{32}$$

assuming constant body force, where the coefficients $\mathbf{a}_1$,$\mathbf{a}_2$,and $\mathbf{a}_3$ are determined from

$$\mathbf{K}_1^{\mathrm{T}} \widetilde{\mathbf{C}}_{11} \mathbf{K}_1 \mathbf{a}_1 = -\mathbf{p}_2 - \mathbf{f}, \tag{33}$$

$$\mathbf{A}_1 \mathbf{a}_1 + \mathbf{C}_{22} \mathbf{a}_2 = -b_1 \mathbf{C}_{12}^{\mathrm{T}} \mathbf{k}_1 - \vartheta \mathbf{C}_{22} \mathbf{k}_3, \tag{34}$$

$$\mathbf{A}_1 \mathbf{a}_2 + \mathbf{C}_{22} \mathbf{a}_3 = -b_2 \mathbf{C}_{12}^{\mathrm{T}} \mathbf{k}_1 + \vartheta \mathbf{C}_{12}^{\mathrm{T}} \mathbf{k}_2. \tag{35}$$

Thus, the general solution of Eq. (18) is

$$\mathbf{u} = 2\mathrm{Re}\left\{\sum_{k=1}^{5} \mathbf{U}_k F_k(z_k)\right\} + \mathbf{a}_1 x_1^2/2 + \mathbf{a}_2 x_1 x_2 + \mathbf{a}_3 x_2^2/2 + \hat{\mathbf{u}}, \tag{36}$$

$$\begin{aligned} \boldsymbol{\tau}_1 = {} & 2\mathrm{Re}\left\{\sum_{k=1}^{5} (\mathbf{A}_4 + p_k \mathbf{C}_{12}) \mathbf{U}_k F_k'(z_k)\right\} + \varepsilon \mathbf{C}_{11} \mathbf{k}_1 - \vartheta \widetilde{\mathbf{C}}_{11} \mathbf{k}_2 x_2 \\ & + \widetilde{\mathbf{C}}_{11}[(\mathbf{K}_1 \mathbf{a}_1 + b_1 \mathbf{k}_1) x_1 + (\mathbf{K}_1 \mathbf{a}_2 + b_2 \mathbf{k}_1) x_2], \end{aligned} \tag{37}$$

$$\boldsymbol{\tau}_2 = 2\mathrm{Re}\left\{\sum_{k=1}^{5} (\mathbf{A}_1 + p_k \mathbf{C}_{22}) \mathbf{U}_k F_k'(z_k)\right\} + \varepsilon \mathbf{C}_{12}^{\mathrm{T}} \mathbf{k}_1, \tag{38}$$

in which

$$\hat{\mathbf{u}} = \begin{bmatrix} -b_1 x_3^2/2 - \vartheta x_2 x_3 & -b_2 x_3^2/2 + \vartheta x_1 x_3 & (b_1 x_1 + b_2 x_2 + \varepsilon) x_3 & 0 & 0 \end{bmatrix}^{\mathrm{T}}.$$

The complex functions $F_k(z_k)$ in Eqs. (36)–(38) for a specific problem are to be determined using the prescribed boundary conditions.

## 4. Electromagnetoelastic field in a half-space

The solution for the problem of an anisotropic elastic half-space under the line loads was given in Section 28 of Lekhnitskii (1981). Here we consider the electromagnetoelastic field in a half-space subjected to line loads and prescribed electromagnetic conditions on the half-space boundary. Lekhnitskii's solution can be carried over to the present case in a simple manner without working with the individual equations and variables.

The mechanical boundary conditions of the problem are

$$\sigma_{22} = N(x_1), \quad \sigma_{12} = T(x_1), \quad \sigma_{23} = 0 \quad \text{on } x_2 = 0, \tag{39}$$

where $N(x_1)$ and $T(x_1)$ are the prescribed normal force and shearing force.

In the absence of electric charges and currents, the electromagnetic boundary conditions require that either the normal component of the electric charge or the electric potential and either the normal magnetic flux or the magnetic potential be prescribed on the boundary

$$\text{either } D_2 = f_{\mathrm{e}}(x_1) \quad \text{or} \quad \phi = g_{\mathrm{e}}(x_1) \quad \text{on} \quad x_2 = 0, \tag{40}$$

$$\text{either } B_2 = f_{\mathrm{m}}(x_1) \quad \text{or} \quad \varphi = g_{\mathrm{m}}(x_1) \quad \text{on} \quad x_2 = 0, \tag{41}$$

where $f_{\mathrm{e}}(x_1)$, $f_{\mathrm{m}}(x_1)$, $g_{\mathrm{e}}(x_1)$ and $g_{\mathrm{m}}(x_1)$ are prescribed functions of $x_1$.

The boundary conditions specified by Eqs. (39)–(41) expressed in the present context are

$$\boldsymbol{\tau}_2(x_1,0)=\mathbf{f}(x_1), \tag{42}$$

$$\mathbf{K}_t\boldsymbol{\tau}_2(x_1,0)=\mathbf{g}_t(x_1),\mathbf{K}_{\mathrm{e}}\mathbf{u}(x_1,0)=\mathbf{g}_u(x_1), \tag{43}$$

where

$$\mathbf{f}(x_1)=[\,T(x_1)\quad N(x_1)\quad 0\quad f_{\mathrm{e}}(x_1)\quad f_{\mathrm{m}}(x_1)\,]^{\mathrm{T}},$$
$$\mathbf{g}_t(x_1)=[\,T(x_1)\quad N(x_1)\quad 0\,]^{\mathrm{T}},\quad \mathbf{g}_u(x_1)=[\,g_{\mathrm{e}}(x_1)\quad g_{\mathrm{m}}(x_1)\,]^{\mathrm{T}},$$
$$\mathbf{K}_t=\begin{bmatrix}1&0&0&0&0\\0&1&0&0&0\\0&0&1&0&0\end{bmatrix},\quad \mathbf{K}_{\mathrm{e}}=\begin{bmatrix}0&0&0&1&0\\0&0&0&0&1\end{bmatrix}.$$

First, we consider the boundary condition given by Eq. (42). Substituting Eq. (38) into Eq. (42) gives

$$2\mathrm{Re}\left\{\sum_{k=1}^{5}(\mathbf{A}_1+p_k\mathbf{C}_{22})\mathbf{U}_kF_k'(x_1)\right\}=\mathbf{f}(x_1)-\varepsilon\mathbf{C}_{12}^{\mathrm{T}}\mathbf{k}_1. \tag{44}$$

Applying the Cauchy integral formula of analytic functions in a half plane (Lekhnitskii, 1981) to Eq. (44) yields a system of five linear algebraic equations for the five unknown $F_k'(z)$ as follows:

$$\sum_{k=1}^{5}(\mathbf{A}_1+p_k\mathbf{C}_{22})\mathbf{U}_kF_k'(z)=-\frac{1}{2\pi i}\int_{-\infty}^{\infty}\frac{\mathbf{f}(x_1)-\varepsilon\mathbf{C}_{12}^{\mathrm{T}}\mathbf{k}_1}{x_1-z}\mathrm{d}x_1. \tag{45}$$

Next, we consider the boundary condition given by Eq. (43). Substitution of Eqs. (36) and (38) in Eq. (43) gives

$$2\mathrm{Re}\left\{\sum_{k=1}^{5}\mathbf{K}_t(\mathbf{A}_1+p_k\mathbf{C}_{22})\mathbf{U}_kF_k'(x_1)\right\}=\mathbf{h}_t(x_1), \tag{46}$$

$$2\mathrm{Re}\left\{\sum_{k=1}^{5}\mathbf{K}_{\mathrm{e}}\mathbf{U}_kF_k(x_1)\right\}=\mathbf{h}_u(x_1), \tag{47}$$

where

$$\mathbf{h}_t(x_1)=\mathbf{g}_t(x_1)-\varepsilon\mathbf{K}_t\mathbf{C}_{12}^{\mathrm{T}}\mathbf{k}_1,\quad \mathbf{h}_u(x_1)=\mathbf{g}_u(x_1)-\mathbf{K}_{\mathrm{e}}\mathbf{a}_1x_1^2/2.$$

Applying the Cauchy integral formula to Eqs. (46) and (47) yields

$$\sum_{k=1}^{5}\mathbf{K}_t(\mathbf{A}_1+p_k\mathbf{C}_{22})\mathbf{U}_kF_k'(z)=-\frac{1}{2\pi i}\int_{-\infty}^{\infty}\frac{\mathbf{h}_t(x_1)}{x_1-z}\mathrm{d}x_1, \tag{48}$$

$$\sum_{k=1}^{5}\mathbf{K}_{\mathrm{e}}\mathbf{U}_kF_k(z)=-\frac{1}{2\pi i}\int_{-\infty}^{\infty}\frac{\mathbf{h}_u(x_1)}{x_1-z}dx_1. \tag{49}$$

Observing that the unknowns in Eq. (48) are $F_k'(z)$, whereas the unknowns in Eq. (49) are $F_k(z)$, we differentiate Eq. (49) with respect to $z$ to obtain

$$\sum_{k=1}^{5}\mathbf{K}_{\mathrm{e}}\mathbf{U}_kF_k'(z)=-\frac{1}{2\pi i}\int_{-\infty}^{\infty}\frac{\mathbf{h}_u(x_1)}{(x_1-z)^2}\mathrm{d}x_1. \tag{50}$$

Eq. (48) and (50) constitute a set of five algebraic equations for the five unknowns $F'_k(z)$. Upon solving for $F'_k(z)$ from Eq. (45) or from Eqs. (48) and (50), depending on the boundary condition being considered, integrating $F'_k(z)$ with respect to $z$ and replacing the variable $z$ by $z_k$, we obtain the analytic functions $F_k(z_k)$ for the problem. All the field variables follow from Eqs. (36)–(38) subsequently.

We note in passing that the displacement boundary condition can be considered in a similar fashion. The solutions for the elastic and piezoelectric counterparts (Lekhnitskii, 1981; Tarn, 2002c) are reproduced by setting $e_{ij} = q_{ij} = 0$, $k = 1$–$3$; and $q_{ij} = \mu_{ij} = 0$, $k = 1$–$4$, respectively.

## 5. Electromagnetoelastic field in a notched plate

Disturbance of the uniform field in an infinite plate by the presence of a notch is a classical problem of anisotropic elasticity (Savin, 1961; Lekhnitskii, 1981). Here we consider the electromagnetoelastic field in a notched plate under uniform extension.

When an infinite plate is subjected to uniform extension, the internal field is uniform. In the presence of a notch, the uniform field is disturbed. The disturbance can be determined by superposing on the uniform field an internal field derived from an auxiliary problem in which the negative of the traction, the normal electric displacement, and the normal magnetic flux resulted from the uniform field is prescribed on the notch contour. The superposition annihilates the loading on the notch boundary and makes the notch free of external loads, thus producing the solution to the original problem.

Consider the notched plate under uniform extension $\varepsilon_0$ in the $x_1$ direction at infinity. The mechanical boundary conditions on the notch boundary are traction-free. The electromagnetic boundary conditions on the notch boundary are assumed to be electromagnetic insulated such that the normal electric displacement and the normal magnetic flux are zero. The condition at infinity is

$$\varepsilon_{11} = \varepsilon_0, \quad \varepsilon_{ij} = 0 \ (i, j \neq 1), \quad \phi \quad \text{and} \quad \varphi = \text{constant}. \tag{51}$$

The uniform field in an infinite plate under uniform extension is

$$[\sigma_{11} \quad \sigma_{22} \quad \sigma_{33} \quad \sigma_{23} \quad \sigma_{13} \quad \sigma_{12}] = \varepsilon_0[c_{11} \quad c_{12} \quad c_{13} \quad c_{14} \quad c_{15} \quad c_{16}], \tag{52}$$

$$[D_1 \quad D_2 \quad D_3 \quad B_1 \quad B_2 \quad B_3] = \varepsilon_0[e_{11} \quad e_{21} \quad e_{31} \quad q_{11} \quad q_{21} \quad q_{31}], \tag{53}$$

which, in the present context, are given by

$$\boldsymbol{\tau}_1 = \varepsilon_0[c_{15} \quad c_{11} \quad c_{13} \quad e_{11} \quad e_{31} \quad q_{11} \quad q_{31}]^{\mathrm{T}}, \tag{54}$$

$$\boldsymbol{\tau}_2 = \varepsilon_0[c_{16} \quad c_{12} \quad c_{14} \quad e_{21} \quad q_{21}]^{\mathrm{T}}. \tag{55}$$

The auxiliary problem requires that the notch boundary be subjected to

$$\begin{bmatrix} t_1 \\ t_2 \\ t_3 \end{bmatrix} = -\varepsilon_0 \begin{bmatrix} c_{11} & c_{16} \\ c_{16} & c_{12} \\ c_{15} & c_{14} \end{bmatrix} \begin{bmatrix} \cos\theta \\ \sin\theta \end{bmatrix}, \quad \begin{bmatrix} D_n \\ B_n \end{bmatrix} = -\varepsilon_0 \begin{bmatrix} e_{11} & e_{21} \\ q_{11} & q_{21} \end{bmatrix} \begin{bmatrix} \cos\theta \\ \sin\theta \end{bmatrix}, \tag{56}$$

where $t_i$, $D_n$, and $B_n$ denote the traction components, the normal electric displacement, and the normal magnetic flux, respectively, $\theta$ is the angle measured counter-clockwise between the $x_1$-axis and the outward normal at a point along the notch boundary.

By using Eqs. (27)–(29), we can express the traction, the normal electric displacement, and the normal magnetic flux along the notch contour in terms of the analytic functions as follows:

$$t_1 = \mathbf{n}_\sigma(\mathbf{K}_1^{\mathrm{T}}\boldsymbol{\tau}_1 + \mathbf{K}_3^{\mathrm{T}}\boldsymbol{\tau}_2) = -2\mathrm{Re}\left\{\sum_{k=1}^{5}\mathbf{n}_\sigma(\mathbf{A}_3 + p_k\mathbf{A}_5)\mathbf{U}_k F_k'(z_k)\right\}, \tag{57}$$

$$t_2 = \mathbf{n}_\sigma\boldsymbol{\tau}_2 = 2\mathrm{Re}\left\{\sum_{k=1}^{5}\mathbf{n}_\sigma(\mathbf{A}_1 + p_k\mathbf{C}_{22})\mathbf{U}_k F_k'(z_k)\right\}, \tag{58}$$

$$t_3 = \mathbf{n}_\sigma(\mathbf{K}_2^{\mathrm{T}}\boldsymbol{\tau}_1 + \mathbf{K}_4^{\mathrm{T}}\boldsymbol{\tau}_2) = 2\mathrm{Re}\left\{\sum_{k=1}^{5}\mathbf{n}_\sigma(\mathbf{A}_6 + p_k\mathbf{A}_7)\mathbf{U}_k F_k'(z_k)\right\}, \tag{59}$$

$$D_n = \mathbf{n}_\mathrm{e}(\mathbf{K}_5^{\mathrm{T}}\boldsymbol{\tau}_1 + \mathbf{K}_7^{\mathrm{T}}\boldsymbol{\tau}_2) = 2\mathrm{Re}\left\{\sum_{k=1}^{5}\mathbf{n}_\mathrm{e}(\mathbf{A}_8 + p_k\mathbf{A}_9)\mathbf{U}_k F_k'(z_k)\right\}, \tag{60}$$

$$B_n = \mathbf{n}_\mathrm{e}(\mathbf{K}_6^{\mathrm{T}}\boldsymbol{\tau}_1 + \mathbf{K}_8^{\mathrm{T}}\boldsymbol{\tau}_2) = 2\mathrm{Re}\left\{\sum_{k=1}^{5}\mathbf{n}_\mathrm{e}(\mathbf{A}_{10} + p_k\mathbf{A}_{11})\mathbf{U}_k F_k'(z_k)\right\}, \tag{61}$$

where

$$\mathbf{K}_5^{\mathrm{T}} = \begin{bmatrix} 0&0&0&0&0&0&0\\ 0&0&0&0&0&0&0\\ 0&0&0&1&0&0&0\\ 0&0&0&0&0&0&0\\ 0&0&0&0&0&0&0 \end{bmatrix}, \quad \mathbf{K}_6^{\mathrm{T}} = \begin{bmatrix} 0&0&0&0&0&0&0\\ 0&0&0&0&0&0&0\\ 0&0&0&0&0&0&0\\ 0&0&0&0&0&1&0\\ 0&0&0&0&0&0&0 \end{bmatrix},$$

$$\mathbf{K}_7^{\mathrm{T}} = \begin{bmatrix} 0&0&0&0&0\\ 0&0&0&0&0\\ 0&0&0&0&0\\ 0&0&0&1&0\\ 0&0&0&0&0 \end{bmatrix}, \quad \mathbf{K}_8^{\mathrm{T}} = \begin{bmatrix} 0&0&0&0&0\\ 0&0&0&0&0\\ 0&0&0&0&0\\ 0&0&0&0&0\\ 0&0&0&0&1 \end{bmatrix},$$

$$\mathbf{n}_\sigma = [\cos\theta \quad \sin\theta \quad 0 \quad 0 \quad 0], \quad \mathbf{n}_\mathrm{e} = [0 \quad 0 \quad 0 \quad \cos\theta \quad \sin\theta],$$
$$\mathbf{A}_5 = \mathbf{K}_1^{\mathrm{T}}\mathbf{C}_{12} + \mathbf{K}_3^{\mathrm{T}}\mathbf{C}_{22}, \quad \mathbf{A}_6 = \mathbf{K}_2^{\mathrm{T}}\mathbf{A}_4 + \mathbf{K}_4^{\mathrm{T}}\mathbf{A}_1,$$
$$\mathbf{A}_7 = \mathbf{K}_2^{\mathrm{T}}\mathbf{C}_{12} + \mathbf{K}_4^{\mathrm{T}}\mathbf{C}_{22}, \quad \mathbf{A}_8 = \mathbf{K}_5^{\mathrm{T}}\mathbf{A}_4 + \mathbf{K}_7^{\mathrm{T}}\mathbf{A}_1,$$
$$\mathbf{A}_9 = \mathbf{K}_5^{\mathrm{T}}\mathbf{C}_{12} + \mathbf{K}_7^{\mathrm{T}}\mathbf{C}_{22}, \quad \mathbf{A}_{10} = \mathbf{K}_6^{\mathrm{T}}\mathbf{A}_4 + \mathbf{K}_8^{\mathrm{T}}\mathbf{A}_1, \ \mathbf{A}_{11} = \mathbf{K}_6^{\mathrm{T}}\mathbf{C}_{12} + \mathbf{K}_8^{\mathrm{T}}\mathbf{C}_{22}.$$

In order to use the boundary conditions to determine the analytic functions for the auxiliary problem, the notch boundary must be expressed properly in terms of the complex variables. This hinges on the existence of the conformal mapping functions that transform the exterior of the notch onto the exterior of a unit circle for all the complex variables $z_k$, $k = 1$–5. It has been shown (Wang and Tarn, 1993) that the conformal mapping in the entire region outside the unit circle is possible only for an elliptic hole in an anisotropic elastic medium. In the present case, the contour of the elliptic hole is transformed onto a unit circle by

$$z_k = m_k\xi_k + \bar{m}_k\xi_k^{-1}, \tag{62}$$

where

$$m_k = (a - \mathrm{i}p_k b)/2, \quad \bar{m}_k = (a + \mathrm{i}p_k b)/2.$$

The inverse relation of Eq. (62) is

$$\xi_k = \frac{z_k + (z_k^2 - 4m_k\bar{m}_k)^{1/2}}{2m_k}. \tag{63}$$

The mapping functions map the exterior of an ellipse

$$\frac{x_1^2}{a^2} + \frac{x_2^2}{b^2} = 1, \quad x_1 = a\cos\theta, \quad x_2 = b\sin\theta \tag{64}$$

for all $z_k$ in the $z$ plane onto the exterior of a unit circle: $\xi = \xi_k = e^{i\theta}$ in the $\xi$ plane, making it possible for us to use the Cauchy integral formula to determine the analytic functions for the problem.

Upon imposing Eq. (56) on Eqs. (57)–(61), multiplying both sides of the equations by $(2\pi i)^{-1}d\xi/(\xi - z)$, integrating them clockwise around the unit circle, applying the Cauchy integral formula of analytic functions for a unit circle in the $\xi$ plane and using

$$F_k'(z) = F_k'(\xi_k)\frac{d\xi_k}{dz_k} = \frac{\xi_k}{m_k\xi_k - \bar{m}_k\xi_k^{-1}}F_k'(\xi_k), \tag{65}$$

there follows:

$$\sum_{k=1}^{5}\boldsymbol{\eta}_\sigma^T(\mathbf{A}_3 + p_k\mathbf{A}_5)\mathbf{U}_kF_k'(z) = -\frac{\varepsilon_0}{\pi}\int_0^{2\pi}\frac{c_{11}\cos\theta + c_{16}\sin\theta}{e^{i\theta} - z}e^{i\theta}d\theta, \tag{66}$$

$$\sum_{k=1}^{5}\boldsymbol{\eta}_\sigma^T(\mathbf{A}_1 + p_k\mathbf{C}_{22})\mathbf{U}_kF_k'(z) = -\frac{\varepsilon_0}{\pi}\int_0^{2\pi}\frac{c_{16}\cos\theta + c_{12}\sin\theta}{e^{i\theta} - z}e^{i\theta}d\theta, \tag{67}$$

$$\sum_{k=1}^{5}\boldsymbol{\eta}_\sigma^T(\mathbf{A}_6 + p_k\mathbf{A}_7)\mathbf{U}_kF_k'(z) = -\frac{\varepsilon_0}{\pi}\int_0^{2\pi}\frac{c_{15}\cos\theta + c_{14}\sin\theta}{e^{i\theta} - z}e^{i\theta}d\theta, \tag{68}$$

$$\sum_{k=1}^{5}\boldsymbol{\eta}_e^T(\mathbf{A}_8 + p_k\mathbf{A}_9)\mathbf{U}_kF_k'(z) = -\frac{\varepsilon_0}{\pi}\int_0^{2\pi}\frac{e_{11}\cos\theta + e_{21}\sin\theta}{e^{i\theta} - z}e^{i\theta}d\theta, \tag{69}$$

$$\sum_{k=1}^{5}\boldsymbol{\eta}_e^T(\mathbf{A}_{10} + p_k\mathbf{A}_{11})\mathbf{U}_kF_k'(z) = -\frac{\varepsilon_0}{\pi}\int_0^{2\pi}\frac{q_{11}\cos\theta + q_{21}\sin\theta}{e^{i\theta} - z}e^{i\theta}d\theta, \tag{70}$$

where

$$\boldsymbol{\eta}_\sigma = \frac{z}{(m_kz - \bar{m}_kz^{-1})}\begin{bmatrix} z^{-1}+z \\ i(z^{-1}-z) \\ 0 \\ 0 \\ 0 \end{bmatrix}, \quad \boldsymbol{\eta}_e = \frac{z}{(m_kz - \bar{m}_kz^{-1})}\begin{bmatrix} 0 \\ 0 \\ 0 \\ z^{-1}+z \\ i(z^{-1}-z) \end{bmatrix}.$$

Eqs. (66)–(70) are five linear algebraic equations for the five unknowns $F_k'(z)$. After solving $F_k'(z)$ and integrating it with respect to $z$ and replacing $z$ by $z_k$, we obtain the analytic functions $F_k(z_k)$ for the auxiliary problem. The auxiliary internal field are determined by substituting $F_k(z_k)$ into Eqs. (27)–(29).

Superposition of the uniform field and the auxiliary internal field yields the electromagnetoelastic field in the infinite plate containing an elliptic notch

$$\boldsymbol{\tau}_1 = \varepsilon_0[c_{15} \quad c_{11} \quad c_{13} \quad e_{11} \quad e_{31} \quad q_{11} \quad q_{31}]^T + 2\text{Re}\left\{\sum_{k=1}^{5}(\mathbf{A}_4 + p_k\mathbf{C}_{12})\mathbf{U}_kF_k'(z_k)\right\}, \tag{71}$$

$$\boldsymbol{\tau}_2 = \varepsilon_0[c_{16} \quad c_{12} \quad c_{14} \quad e_{21} \quad q_{21}]^T + 2\text{Re}\left\{\sum_{k=1}^{5}(\mathbf{A}_1 + p_k\mathbf{C}_{22})\mathbf{U}_kF_k'(z_k)\right\}. \tag{72}$$

Again, by setting $e_{ij} = q_{ij} = 0$, $k = 1$–3, and $q_{ij} = \mu_{ij} = 0$, $k = 1$–4, respectively, the solutions of the associated problems of anisotropic elasticity (Lekhnitskii, 1981) and piezoelectricity (Tarn, 2002c) are recovered.

In closing, we note that a series solution for the problem may be found, following the same line as given in Lekhnitskii (1981), by representing the analytic functions in the exterior of a unit circle in the $\xi$ plane in the form of Laurent's series

$$F_k(\xi) = b_k \ln \xi + \sum_{n=0}^{\infty} a_{nk}\xi^{-n}, \quad \xi = \mathrm{e}^{\mathrm{i}\theta}, \tag{73}$$

in which $b_k$ and $a_{nk}$ are determined by comparing the coefficients on both sides of the algebraic equations resulting from the notch boundary conditions.

## 6. Concluding remarks

We have formulated the basic equations of static electromagnetoelasticity in the state space setting and derived a state equation and an output equation that bear a remarkable resemblance to their elastic counterparts, making it possible to solve various stationary problems of electromagnetoelasticity in parallel to the associated problems of anisotropic elasticity. For illustration, we have determined the exact solutions for two classes of problems by simple extension of the corresponding elastic solutions. Other problems of electromagnetoelasticity can be treated as well following the same line.

The connection between static electromagnetoelasticity and anisotropic elasticity has been found by grouping the field variables properly and partitioning the constitutive matrix accordingly. In the formulation we have grouped the field variables in such a manner that the derivatives with respect to $x_2$ are taken to the left-hand side of the state equation. The way of grouping is not unique. For other groupings the forms of the state equation and output equation remain unchanged, only the matrices $\mathbf{C}_{\alpha\beta}$ and $\mathbf{K}_i$ need to be redefined. It has been shown in the state space formalism for anisotropic elasticity (Tarn, 2002a) that an alternative formulation based on grouping the stresses into inplane and antiplane components results in a state equation and an output equation different in form but same in effect. The statement stands in the present case.

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